(For candidates admitted from 2016–2017 onwards)

M.Sc. DEGREE EXAMINATION, APRIL 2023.

Mathematics

MEASURE AND INTEGRATION

Time : Three hours Maximum : 75 marks

SECTION A — ($10 \times 2 = 20$)

Answer ALL questions.

1. Prove that outer measure is translation invariant.

2. Define a measurable function.

3. For any non-negative measurable function f, define the integral of f.

4. Show that if f and g are measurable, $|f| \leq |g|$ a.e., and g is integrable, then show that f is integrable.

5. When do we say that a given ring is a σ – ring?

6. Show that Lebesgue measure m_* defined on $\mathcal{M}$, the class of measureable sets of $\mathbb{R}$, is σ – finite and complete.

7. When a sequence of functions are said to he fundamental?

8. Prove that uniform convergence a.e. implies almost uniform convergence.

9. Define a measurable rectangle.

10. Define monotone class of a space.

SECTION B — ($5 \times 5 = 25$)

Answer ALL questions, choosing either (a) or (b).

11. (a) Prove that every Borel set is measurable.

Or

(b) Show that there exist uncountable set of zero measure.

12. (a) Show that if f is a non-negative measurable function, then $f = 0$ a.e. if. and only if, $\int f dx = 0$.

Or

(b) State and prove Lebesgue dominated convergence theorem.

13. (a) Show that every algebra is a ring and every σ-algebra a σ-ring but that the converse is not true.

Or

(b) Show that the completion of a σ-finite measure is σ-finite.

14. (a) If a sequence of measurable functions converges in measure, then show that the limit function is unique a.e.

Or

(b) Let $\{f_n\}$ be a sequence of norm-negative measurable functions and let f be a measurable function such that $f_n \to f$ in measure: then show that $\int f d\mu \leq \liminf \int f_n d\mu$.

15. (a) If y is any class of subsets of X, then show that there exists a smallest monotone class, denoted by $\mathcal{M}_0(y)$, containing y.

Or

(b) Let f be an $\mathcal{S} \times \mathcal{T}$ measurable function on $X \times Y$; then show that for each $x \in X$ and $y \in Y$, f_x is a $\mathcal{T}$ measurable function and f^y is an S–measurable function.

SECTION C — (3 × 10 = 30)

Answer any THREE questions.

16. Prove that outer measure of an interval equals its length.

17. State and prove Fatou's lemma.

18. If μ is a measure on a σ-ring S, then show that the class $\overline{S}$ of sets of the form $E \Delta N$ for any sets E, N such that $E \in S$ while N is contained in some set in S of zero measure, is a σ-ring, and the set function $\overline{\mu}$ defined by $\overline{\mu}(E \Delta N) = \mu(E)$ is a complete measure on $\overline{S}$.

19. If $\{f_n\}$ is a sequence of measurable functions which is fundamental in measure, then show that there exists a measurable function f such that $f_n \to f$ in measure.

20. State and prove Fubini's theorem.

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