F–3113

Sub. Code
7PMA1C2

M.Phil. DEGREE EXAMINATION, NOVEMBER 2019

First Semester

Mathematics

FUNCTIONAL ANALYSIS

(CBCS – 2017 onwards)

Time : 3 Hours Maximum : 75 Marks

Section A ($5 \times 5 = 25$)

Answer any **five** questions.

1. Define the following terms :

   (a) Vector space

   (b) Topological space

   (c) Hausdorff space

   (d) Linear mapping.

2. Suppose that $(X_1\, d_1)$ and $(Y_1\, d_2)$ are metric spaces , and $(X_1\, d_1)$ is complete. If E is a closed set in X, $f : E \to Y$ is continuous, and $d_2(f(x'), f(x'') \geq d_1(x', x''))$ for all $x', x'' \in E$, then prove that $f(E)$ is closed.

3. Suppose A is a convex absorbing set in a vector space X. Prove that

   (a) $\mu_A(x+y) \leq \mu_A(x) + \mu_A(y)$

   (b) $\mu_A(tx) = t\,\mu_A(x)$ if $f \geq 0$.

4. Write the usual notations, prove that L^p is a locally bounded F-space.

5. State and prove the Baire's theorem.

6. Suppose M is a subspace of a vector X, P is a seminorm on X, and f is a linear functional on M such that $|f(x)| \leq p(x)$ $(x \in M)$. Prove that p extends to a linear functional $\wedge$ on X that satisfies $|\wedge x| \leq p(x)\,(x \in X)$.

7. If X and Y are normed spaces and if $\wedge \in \mathfrak{B}\,(X,Y)$, then prove that $\|\wedge\| = \sup\left\{\left|< \wedge x, y^* > 1\right| : \|x\| \leq 1, \left\|y^*\right\| \leq 1\right\}$.

8. Define the following terms :

   (a) Compact

   (b) Invertible

   (c) Spectrum of an operator

   (d) Eigen value

   (e) Direct sum.

Section B (5 × 10 = 50)

Answer **all** questions, choosing either (a) or (b).

9. (a) Let $\wedge$ be a linear functional on a topological vector space X. Assume $\wedge x \neq 0$ for some $x \in X$. Prove that each of the following four properties implies the other three.

   (i) $\wedge$ is continuous

   (ii) the null space $\mathscr{N}(\wedge)$ is closed

   (iii) $\mathscr{N}(\wedge)$ is not dense in X

   (iv) $\wedge$ is bounded in some neighborhood $\vee$ of 0.

Or

(b) (i) If X is a complex topological vector space and $f : \mathbb{C}^n \to X$ is linear, then prove that f is continuous.

(ii) Show that every locally compact topological vector space X has finite dimension.

10. (a) (i) Define Cauchy sequence. Also prove that sequence $\{x_n\}$ in X is a d-Cauchy sequence if and only if it is a τ - Cauchy sequence.

(ii) Define the following terms.

Bounded linear transformation, Seminorm, Minkouski functional μ_A, Quotient space of X modulo N ,Quotient topology.

Or

(b) (i) Prove that a topology vector space X is normable if and only if its origin has a convex bounded neighborhood.

(ii) Show that $C(\Omega)$ is a Frechet space.

11. (a) Suppose

(i) X is an F-space

(ii) Y is a topological vector space,

(iii) $\wedge : X \to Y$ is continuous and linear and

(iv) $\wedge(X)$ is of the second category in Y. Prove that the following:

(1) $\wedge(X) = Y$

(2) $\wedge$ is an open mapping

(3) Y is an F-space.

Or

(b) (i) State the Banach-Steinhaus theorem.

(ii) Define the following terms:

Bilinear mapping and separately continuous.

(iii) State and prove the closed graph theorem.

12. (a) State and prove the Banach-Alaoglu theorem.

Or

(b) State and prove Milman's theorem.

13. (a) If X and Y are Banach spaces and if $T \in \mathscr{B}(X,Y)$, then prove the following three conditions implies the other two

(i) $\mathscr{R}(T)$ is closed in Y

(ii) $\mathscr{R}(T^*)$ is week*-closed in X^*;

(iii) $\mathscr{R}(T^*)$ is norm-closed in X^*.

Or

(b) Suppose X is a Banach space, $T \in \mathscr{B}(X)$, and T is compact. Prove the following :

(i) If $\lambda \neq 0$, then the four numbers

$\alpha = \dim \mathscr{N}(T - \lambda I)$, $\beta = \dim X / \mathscr{R}(T - \lambda I)$ are equal and finite.

(ii) If $\lambda \neq 0$ and $\lambda \in \sigma(T)$ then λ is an eigen value of T and of T^*.

(iii) $\sigma(T)$ is compact, at most countable, and has at most one limit point, namely 0.

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